HW7
Math 184A
Winter 2008

John Kordich (A03-69-9873)

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4.3

2.)

Let S be a set with a permutation group G. According to Burnside's Lemma, the number of equivalence classes that G defines on S is

$$\frac{1}{|G|}\sum_{g\in G} N(g) \tag{1}$$

where $N(g)$ is the number of $x \in S$ such that $g(x) = x$.

For this problem of 8-long circular sequences made of the set $\{0,1,2,3,4,5,6,7,8,9\}$, let us define 8 rotations, $G = \{g_0, g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$, where for any integer $0 \le i \le 7$ and any 8-long sequence, $g_i(x)$ rotates the sequence x to the left i spaces.

Let us begin to set up the problem to work with Burnside's Lemma. Clearly, $|G| = 8$. Now we need to find all $N(g_i)$.
$N(g_0)$ = the size of S. No digit can be used more than once, so this is just a basic list without repeats. Therefore,

$$N(g_0) = \frac{10!}{(10-8)!} = \frac{10!}{2!} = 1814400$$

$N(g_1)$ = the number of elements of S that will satisfy $g_1(x) = x$. Since each digit in an element of S must be different than the other digits in that element (each are picked only once), there are no elements of S that satisfy $g_1(x) = x$. Therefore,

$$N(g_1) = 0$$

This is the same for all other g_i for $1 \le i \le 7$.

$$N(g_1) = N(g_2) = N(g_3) = N(g_4) = N(g_5) = N(g_6) = N(g_7) = 0.$$

Thus, the total number of 8-long circular sequences that can be made from $\{0,1,2,3,4,5,6,7,8,9\}$ if no digit is used more than once is:

$$= \frac{1}{8}\sum_{g\in G} N(g)$$

$$= \frac{1}{8}N(g_0)$$

$$\boxed{= 226800}$$

Now, to generalize the answer to n-long circular sequences when k things are available instead of just ten, this is quite easy. (Assuming no digit can appear more than once still). If no digit can appear more than once, then the total number of n-long circular sequences that can be made from k-things is:

$$\boxed{= \frac{1}{n}\frac{k!}{(k-n)!}}$$

6.)

a.

We will use Burnside's lemma again here. Using the notes from the homework assignment, we will assume each square has a color, either white or green. Since this is a 4 × 4 board of 16 squares, the set G consists only of 4 rotations. Let $G = g_0, g_1, g_2, g_3$ and let each g_i correspond to a rotation of $90°i$.

Now let's calculate $N(g_i)$.

$N(g_0)$ = the total possible combinations of 8 green and 8 white squares on a 16 piece board. This will be:

$$N(g_0) = \binom{16}{8} = \frac{16!}{8! \cdot 8!} = 12870.$$

Now, for $N(g_1)$, let us consider what squares must be green so that $g_1(x) = x$, where x is a 4 × 4 board composition with 8 green and 8 white squares. First, let us define the sequence P such that $P_i \in \{G, W\}$ and $0 \le i \le 15$. This board maps the sequence onto the 4 × 4 16 piece board.

P_0	P_1	P_2	P_3
P_4	P_5	P_6	P_7
P_8	P_9	P_{10}	P_{11}
P_{12}	P_{13}	P_{14}	P_{15}

In order for $g_1(x) = x$, the following conditions must be true:

$P_0 = P_3 = P_{15} = P_{12}$
$P_4 = P_2 = P_{11} = P_{13}$
$P_5 = P_6 = P_{10} = P_9$
$P_1 = P_7 = P_{14} = P_8$

Also, 8 of $P_i = G$ and 8 of $P_i = W$. Choosing one location as green automatically chooses 3 other locations as green, and the same for white. Therefore, there are 4 places to choose and 2 choices to make, thus:

$$N(g_1) = \binom{4}{2} = \frac{4!}{2! \cdot 2!} = 6$$

Because of symmetry, g_3 has the same requirements as g_1 and therefore,

$N(g_3) = N(g_1) = 6$

Now, to find $N(g_2)$, we must rewrite the rules.

$P_0 = P_{15}$
$P_1 = P_{14}$
$P_5 = P_{10}$
$P_4 = P_{11}$
and
$P_{12} = P_3$
$P_{13} = P_2$

$P_8 = P_7$
$P_9 = P_6$

Again, 8 must be G, 8 must be W. Chooseing 1 location automatically chooses 1 other location. Therefore, there are 8 places to choose and 4 choices to make, thus:

$$N(g_2) = \binom{8}{4} = \frac{8!}{4! \cdot 4!} = 70$$

Now that we have all of $N(g_i)$ we can find the total number of ways 8 squares can be colored green on a 4×4 board of 16 squares where rotated configurations are considered equivalent:

$$= \frac{1}{4}\left[N(g_0) + N(g_1) + N(g_2) + N(g_3)\right]$$

$$\boxed{= 3238}$$

b.

In this part, we will also include flipping the board over. We will use the same rotations functions as in part a, but we will also use flipping functions. There will be 4 flipping functions, as defined here:

g_h which will be a flip over the horizontal line passing through the middle
g_v which will be a flip over the vertical line passing through the middle
g_{tlbr} which will be a flip over the diagonal line passing through the top left and bottom right
g_{trbl} which will be a flip over the diagonal line passing through the top right and bottom left

To find $N(g_h)$ we shall consider which squares need to stay the same:

$P_0 = P_{12}$
$P_1 = P_{13}$
$P_2 = P_{14}$
$P_3 = P_{15}$
$P_4 = P_8$
$P_5 = P_9$
$P_6 = P_{10}$
$P_7 = P_{11}$

Again, this means:

$$N(g_h) = \binom{8}{4} = 70$$

For vertical, there will be just as many for $N(g_v)$, the requirements are different but they follow similar symmetry.

$$N(g_v) = N(g_h) = 70$$

Now, let's look at g_{tlbr}. This would require these equivalences:

$P_4 = P_1$
$P_8 = P_2$
$P_9 = P_6$
$P_{12} = P_3$
$P_{13} = P_7$
$P_{14} = P_{11}$
and now the static positions in this flip:
$P_0 = P_0$
$P_5 = P_5$
$P_{10} = P_{10}$
$P_{15} = P_{15}$

Since there must be 8 green squares and 8 white squares, the static positions must either have 0 green squares, 2 green squares, or 4 green squares. If they had 1 or 3 green squares, the other squares on the board could not make the total number of squares 8 because it would be an odd number and they can only contribute 2 squares at a time. Thus,

$$N(g_{tlbr}) = \binom{4}{0}\binom{6}{4} + \binom{4}{2}\binom{6}{3} + \binom{4}{4}\binom{6}{2} = 15 + 120 + 15 = 150$$

And for symmetry reasons, $N(g_{trbl})$ is the same.

$$N(g_{tlbr}) = N(g_{trbl}) = 150$$

Now that we have all of $N(g)$ we can find the total number of ways 8 squares can be colored green on a 4×4 board of 16 squares where rotated and flipped configurations are considered equivalent:

$$= \frac{1}{8}\left[N(g_0) + N(g_1) + N(g_2) + N(g_3) + N(g_h) + N(g_v) + N(g_{tlbr}) + N(g_{trbl})\right]$$

$$\boxed{= 1674}$$

H

H-10

a.

$f(x) = 2x$ and $x \in \underline{5}$.

The two line notation of this is:
$$\boxed{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}}$$

b.

$\underline{3}^{\underline{2}}$ is equivalent to the set of all functions $f : \underline{2} \to \underline{3}$
And these are the set of functions in two line notation:

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}$$

c.

The functions:

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

are injective only.

The functions:

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}$$

are neither injective nor surjective (and therefore not bijective).

d.

To compute the number of surjections on $f : \underline{100} \to \underline{3}$ we use equation 4.5 from section 4.1 in the book:

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n \tag{2}$$

where $n = 100$ and $k = 3$.

Therefore, the number of surjections on f is:

$$\left[(-1)^0 \binom{3}{0} (3-0)^{100} + (-1)^1 \binom{3}{1} (3-1)^{100} + (-1)^2 \binom{3}{2} (3-2)^{100} + (-1)^3 \binom{3}{3} (3-3)^{100}\right]$$
$$\boxed{= 3^{100} - 3 \cdot 2^{100} + 3}$$
which is $\approx 5.154 \times 10^{47}$

e.

The stirling number S(100,3) is going to be the same as part d, except with a $\frac{1}{k!}$ coeffcient multiplied to it.

Therefore, S(100,3) $= \frac{1}{3!}\left(3^{100} - 3 \cdot 2^{100} + 3\right)$

$\boxed{= \frac{1}{6}\left(3^{100} - 3 \cdot 2^{100} + 3\right)} \approx 8.5896 \times 10^{46}$

H-11

a.

Since it is a ferris wheel, we'll deal with the set of 6 rotations $G = \{g_0, g_1, g_2, g_3, g_4, g_5\}$ where each g_i for $0 \leq i \leq 5$ rotates the farris wheel counterclockwise by $60°i$. Also, let us define the sequence P such that $P_i \in \{B, W, R\}$ and $0 \leq i \leq 5$.

As with earlier, we'll use Burnside's Lemma.

$N(g_0) =$ all possible combinations without rotations $= \binom{3}{1}^6 = 3^6 = 729$

For $N(g_1)$, $g_1(x) = x$ only when all the colors on the farris wheel are the same. (i.e. $P_0 = P_1 = P_2 = P_3 = P_4 = P_5$)
Therefore,

$N(g_1) = 3$

For $N(g_2)$, $g_2(x) = x$ when

$$P_0 = P_2 = P_4$$
$$P_1 = P_3 = P_5$$

We have 2 choices, each from a 3 element set.
Therefore,

$N(g_2) = 3 \cdot 3 = 9$

For $N(g_3)$, $g_3(x) = x$ when

$$P_0 = P_3$$
$$P_1 = P_4$$
$$P_2 = P_5$$

We have 3 choices, each from a 3 element set.
Therefore,

$N(g_3) = 3^3 = 27$

For $N(g_4)$ and $N(g_5)$ we have similar symmetrys to$N(g_2)$ and $N(g_1)$ respectively. (Imagine reverse rotations)
Therefore,

$N(g_4) = N(g_2) = 9$
$N(g_5) = N(g_1) = 3$

Now we can calculate the total colorings of ferris wheels on 6 seats for 3 possible colors for each seat:

$= \frac{1}{6}[729 + 3 + 9 + 27 + 9 + 3]$

$\boxed{= 130}$

b.

I will start doing this without so much explanation now.

$N(g_0) = \binom{6}{2}\binom{4}{2}\binom{2}{2} = 90$

$N(g_1) = 0$ since all the colors must be the same, it is impossible.

$N(g_2) = 0$ since 3 seats must have the same color, it is impossible.

$N(g_3) = 3! = 6$ since there are 3 (pairs of) seats that must be chosen, each a different color.

$N(g_4) = 0$

$N(g_5) = 0$

Now we can calculate the total colorings of ferris wheels on 6 seats for 2 seats black, 2 seats white, 2 seats red:

$= \frac{1}{6}[90 + 0 + 0 + 6 + 0 + 0]$

$\boxed{= 16}$

c.

Orb(RBWRBW) = {(RBWRBW), (BWRBWR), (WRBWRB), (WBRWBR), (BRWBRW), (RWBRWB)}

d.

Orb(RBWRBW) = {(BRWBRW), (BWRBWR), (RBWRBW), (RWBRWB), (WBRWBR), (WRBWRB)}

e.

8-bead necklace

Stab(RBRWRBRW) = $\boxed{\{1, \rho^4\}}$

H-12

a.

This problem will reduce to the calculation of the number of simple graphs with n vertices. There are $2^{\binom{n}{2}}$ simple graphs for n vertices. Let $S = V, E$ be a simple graph with $V = \underline{n}$ vertices. There are $\binom{n}{2}$ possible edges for a simple graph with n vertices.

Let $S = \{V, E\}$ and $|V| = n$. Now, for every edge $e = \{u, v\}$ where $u \neq v$, we can choose to either have the edge point to u or to v. Therefore, for every simple graph with $|V| = n$, you can derive $2^{|E|}$ oriented simple graphs. For a graph that has n vertices, there are $\binom{\binom{n}{2}}{q}$ graphs that have q edges. Therefore,

The total number of oriented simple graphs with n vertices is:

$$= \sum_{i=0}^{\binom{n}{2}} 2^i \binom{\binom{n}{2}}{i}$$

$$\boxed{= 3^{\binom{n}{2}}}$$

b.

This was (partly) derived in part a. It is:

$$\boxed{= 2^k \binom{\binom{n}{2}}{k}}$$

c.

Let $V = \{1, 2, ..., 10\}$

$$a_k = 2^k \binom{\binom{n}{2}}{k}$$

$$\rightarrow A(x) = \sum_{i=0}^{\infty} 2^i x^i \binom{\binom{n}{2}}{i}$$

$$= \sum_{i=0}^{\binom{n}{2}} (2x)^i \binom{\binom{n}{2}}{i}$$

Using the binomial theorem,

$$\boxed{\rightarrow A(x) = (1 + 2x)^{\binom{n}{2}}}$$